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Almost-commutative geometries beyond the standard model

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Abstract

In Iochum *et al* (2004 *J. Math. Phys.* **45** 5003), Jureit and Stephan (2005 *J. Math. Phys.* **46** 043512), Schücker T (2005 *Preprint* hep-th/0501181) and Jureit *et al* (2005 *J. Math. Phys.* **46** 072303), a conjecture is presented that almost-commutative geometries, with respect to sensible physical constraints, allow only the standard model of particle physics and electro-strong models as Yang–Mills–Higgs theories. In this paper, a counter-example will be given. The corresponding almost-commutative geometry leads to a Yang–Mills–Higgs model which consists of the standard model of particle physics and two new fermions of opposite electro-magnetic charge. This is the second Yang–Mills–Higgs model within noncommutative geometry, after the standard model, which could be compatible with experiments. Combined to a hydrogen-like composite particle, these new particles provide a novel dark matter candidate.

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1. Introduction

The origin of the internal symmetries of the standard model of particle physics, i.e. its gauge group $G_{\mathrm{SM}} = U(1)_Y \times SU(2)_w \times SU(3)_C$, has been one of the most prominent mysteries of physics. While the spacetime symmetries associated with general relativity have a clear geometrical interpretation as the diffeomorphisms of the spacetime manifold, such an interpretation was lacking for the standard model. A further long-standing problem was the unification of these two apparently so different symmetries under one single mathematical roof. This unification is to be understood in the sense of having a manifold-like object which exhibits both internal and spacetime symmetries. As a solution to this unification problem, two main propositions have been made. The first and most widely known is supersymmetry. The second, going back to Heisenberg, inherits the idea that spacetime itself should lose its meaning when high energies are involved. In the spirit of the uncertainty relation, spacetime should become *noncommutative* in the sense that the notion of a spacetime point loses its meaning, in analogy with phase space in quantum mechanics. The original intention was to get rid of unwanted infinities in quantum field theory. Later, it turned out that for classical field theories these noncommutative spaces could be used to unify the inner and spacetime symmetries since one is no longer restricted to symmetry groups of ordinary manifolds.

In the last few years, several approaches to include the idea of noncommutative spaces into physics have been established. One of the most promising and mathematically elaborate is Connes' *Noncommutative Geometry* [5] where the main idea is to translate the usual notions of manifolds and differential calculus into an algebraic language. Instead of describing manifolds by atlases, charts and differential structures, the geometric information is translated into a set of algebraic entities which have to meet a number of axioms. These sets are called *spectral triples* and consist, for the calibrating case of a Riemannian spin manifold, of the commutative involution algebra of C^∞-functions over the manifold faithfully represented on the Hilbert space of square integrable spinors, the usual Dirac operator, the charge conjugation operator and a chirality operator. These elements comply with the axioms of noncommutative geometry given by Connes and allow us to reconstruct all the geometric data of the manifold, such as dimension, geodesic distance and the notion of a point. For a detailed introduction of the main mathematical notions, we refer to [5–8].

A crucial feature of this algebraic approach is that the spectral triples and their axioms are independent of the commutativity of the algebra. It is in fact possible to construct consistent geometries based on matrix algebras, which exhibit in general a noncommutative multiplication, if the remaining elements of the spectral triple are chosen in an appropriate way. Such an algebra is a direct sum of a finite number of simple matrix algebras.

Furthermore, the tensor product of two spectral triples is again a spectral triple and produces a consistent geometry. In the commutative case, the tensor product of spectral triples is equivalent to the direct product of manifolds. The tensor product of the spectral triple of a four-dimensional Riemannian spin manifold and the spectral triple based on a matrix algebra is called an *almost-commutative* spectral triple.

The physical interest of this algebraic description of geometry becomes clear when considering spacetime symmetries and the internal symmetries of the standard model. In noncommutative geometries, the symmetries correspond to the automorphism group of the algebra. It is immediately clear that the commutative algebra of C^∞-functions over a manifold has the diffeomorphisms of the manifold as its automorphism group, i.e. the symmetry group of general relativity. On the other hand, the gauge group $G_{\mathrm{SM}} = U(1)_Y \times SU(2)_w \times SU(3)_C$ of the standard model of particle physics corresponds to the automorphism group of the matrix algebra $\mathcal{A}_{\mathrm{SM}} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, $\mathbb{H}$ being the quaternions. For this so-called *internal* or *finite* geometry, one takes as fermionic Hilbert space that of the standard model and as the Dirac operator the corresponding mass matrix of the fermions. The tensor product of the spectral triples corresponding to these two algebras thus exhibits the combined spacetime and internal symmetries.

Having set up the geometric stage, Connes and Chamseddine defined the so-called *spectral action* [9]. It is given by the number of eigenvalues of the Dirac operator, fluctuated with the lifted automorphisms of the algebra, up to a sufficiently large cut-off. By use of a technique known as heat-kernel expansion, this gives—as leading terms in the cut-off—the desired Einstein–Hilbert Lagrangian plus a cosmological constant and the bosonic part of the standard model Lagrangian. The Higgs potential appears in the correct form and the Higgs scalar acquires a geometric interpretation as the gravitational field in the noncommutative part of spacetime. The Lagrangian for the fermion dynamics and the Yukawa terms are produced by the scalar product on the Hilbert space, which is given automatically. In combination with the spectral action, any almost-commutative spectral triple produces Einsteinian gravity and a Yang–Mills–Higgs theory. For a pedagogical introduction see [10].

In this way, general relativity and the standard model of particle physics become unified as classical field theories. The question that arises immediately is the uniqueness of the choice of the internal algebra, the corresponding Hilbert space and the Dirac operator. To illuminate the structure of almost-commutative geometries from a physicists' point of view, they have been classified by imposing a list of physical requirements for up to four summands in the matrix algebra [1–4]. The Yang–Mills–Higgs theories were asked to exhibit a non-degenerate spectrum of fermion masses, freedom of harmful anomalies and a complex representation of the little group on each fermion multiplet. Furthermore, the Hilbert space was asked to be as small as possible which leads to the restriction of dealing only with one generation of fermions. This requirement translates to the notion of irreducibility for the spectral triples. From the necessity to lift the automorphisms of the algebra to the Hilbert space, combined with the requirement of an anomaly free theory, follow the charges of the fermions with respect to the gauge group [11]. The essence of this classification is a prominent position of the standard model of particle physics within a small set of models, up to an arbitrary number of colours. Apart from the so-called *electro-strong* model, it is only the Yang–Mills–Higgs theory which fulfils the physical requirements stated above for less than five summands in the finite part of the algebra.

The classification can essentially be reduced to a classification of the internal spectral triple, taking into account only the Yang–Mills–Higgs part of the heat-kernel expansion. Here, a diagrammatic approach is possible [12, 13]. The so-called *Krajewski diagrams* encode the algebraic data of the matrix part of the spectral triple, up to the size of the individual summands of the matrix algebra and the corresponding Hilbert spaces. Finding the diagrams corresponding to a given number of summands in the matrix algebra of the spectral triple is a tremendous combinatorial task and thus necessitates the use of a computer [14]. For up to four summands the classification is feasible, since only up to 37 Krajewski diagrams have to be examined. But already for five summands at least 18 000 possible Krajewski diagrams exist and a complete classification seems out of reach.

The authors of [1–4] have put forward a conjecture stating that the only Yang–Mills–Higgs models compatible with almost-commutative geometry are either the standard model of particle physics or electro-strong models. In this paper, a counter-example will be presented which includes the standard model as a sub-model and may exhibit interesting phenomenological consequences with respect to dark matter.

2. A short review of finite spectral triples

As was already noted in [1], it is sufficient to focus on the finite part of almost-commutative spectral triples, if one is interested in the particle content of the corresponding Yang–Mills–Higgs theories. Thus, only the necessary axioms and definitions for the internal part of the almost-commutative geometries will be given.

2.1. Basic definitions of noncommutative geometry

In this section, the necessary basic definitions for the finite part of almost-commutative geometries will be given, i.e. the axioms for real, S^0-real, finite spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D}, J, \epsilon, \chi)$. The algebra $\mathcal{A}$ is a finite sum of matrix algebras $\mathcal{A} = \oplus_{i=1}^N M_{n_i}(\mathbb{K}_i)$ with $\mathbb{K}_i = \mathbb{R}, \mathbb{C}, \mathbb{H}$, where $\mathbb{H}$ denotes the quaternions. A faithful representation ρ of $\mathcal{A}$ is given on the finite-dimensional Hilbert space $\mathcal{H}$. The Dirac operator $\mathcal{D}$ is a self-adjoint operator on $\mathcal{H}$ and plays the role of the fermionic mass matrix. J is an antiunitary involution, $J^2 = 1$, and is interpreted as the charge conjugation operator of particle physics. The S^0-real structure ϵ is a unitary involution, $\epsilon^2 = 1$. Its eigenstates with eigenvalue +1 are the particle states, eigenvalue -1 indicates antiparticle states. The chirality χ is as well a unitary involution, $\chi^2 = 1$, whose eigenstates with eigenvalue $+1(-1)$ are interpreted as right (left) particle states. These operators are required to fulfil Connes' axioms for spectral triples:

- $[J, \mathcal{D}] = [J, \chi] = [\epsilon, \chi] = [\epsilon, \mathcal{D}] = 0, \epsilon J = -J\epsilon, \mathcal{D}\chi = -\chi\mathcal{D}$,
  $[\chi, \rho(a)] = [\epsilon, \rho(a)] = [\rho(a), J\rho(b)J^{-1}] = [[\mathcal{D}, \rho(a)], J\rho(b)J^{-1}] = 0, \forall a, b \in \mathcal{A}$.
- The chirality can be written as a finite sum $\chi = \sum_i \rho(a_i)J\rho(b_i)J^{-1}$. This condition is called *orientability*.
- The intersection form $\cap_{ij} := \mathrm{tr}(\chi\rho(p_i)J\rho(p_j)J^{-1})$ is non-degenerate, $\det\cap \neq 0$. p_i are minimal rank projections in $\mathcal{A}$. This condition is called *Poincaré duality*.

Now the Hilbert space $\mathcal{H}$ and the representation ρ decompose with respect to the eigenvalues of ϵ and χ into left and right particle and antiparticle spinors and representations:

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_L^c \oplus \mathcal{H}_R^c, \\ \rho &= \rho_L \oplus \rho_R \oplus \overline{\rho_L^c} \oplus \overline{\rho_R^c}. \end{aligned} \tag{1}$$

In this representation, the Dirac operator has the form

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{M} & 0 & 0 \\ \mathcal{M}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{\mathcal{M}} \\ 0 & 0 & \overline{\mathcal{M}^*} & 0 \end{pmatrix}, \tag{2}$$

where $\mathcal{M}$ is the fermionic mass matrix connecting the left- and right-handed fermions.

Since the individual matrix algebras have only one fundamental representation for $\mathbb{K} = \mathbb{R}, \mathbb{H}$ and two for $\mathbb{K} = \mathbb{C}$ (the fundamental one and its complex conjugate), ρ may be written as a direct sum of these fundamental representations with multiplicities

$$\rho\big(\oplus_{i=1}^N a_i\big) := \big(\oplus_{i,j=1}^N a_i \otimes 1_{m_{ji}} \otimes 1_{(n_j)}\big) \oplus \big(\oplus_{i,j=1}^N 1_{(n_i)} \otimes 1_{m_{ji}} \otimes \overline{a_j}\big).$$

The first summand denotes the particle sector and the second the antiparticle sector. For the dimensions of the unity matrices, we have $(n) = n$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $(n) = 2n$ for $\mathbb{K} = \mathbb{H}$ and the convention $1_0 = 0$. The multiplicities m_{ji} are non-negative integers. Acting with the real structure J on ρ permutes the main summands and complex conjugates them. It is also possible to write the chirality as a direct sum

$$\chi = \big(\oplus_{i,j=1}^N 1_{(n_i)} \otimes \chi_{ji} 1_{m_{ji}} \otimes 1_{(n_j)}\big) \oplus \big(\oplus_{i,j=1}^N 1_{(n_i)} \otimes \chi_{ji} 1_{m_{ji}} \otimes 1_{(n_j)}\big),$$

where $\chi_{ji} = \pm 1$ according to the previous convention on left-handed (right-handed) spinors. One can now define the multiplicity matrix $\mu \in M_N(\mathbb{Z})$ such that $\mu_{ji} := \chi_{ji} m_{ji}$. This matrix is symmetric and decomposes into a particle and an antiparticle matrix, the second being just the particle matrix transposed, $\mu = \mu_P + \mu_A = \mu_P + \mu_P^T$. The intersection form of the Poincaré duality is now simply $\cap = \mu + \mu^T$, see [12, 13].

The mass matrix $\mathcal{M}$ of the Dirac operator connects the left- and right-handed fermions. Using the decomposition of the representation ρ and the corresponding decomposition of the Hilbert space $\mathcal{H}$, we find two types of submatrices in $\mathcal{M}$, namely $M \otimes 1_{(n_k)}$ and $1_{(n_k)} \otimes M$. M is a complex $(n_i) \times (n_j)$ matrix connecting the ith and the jth sub-Hilbert space and its eigenvalues give the masses of the fermion multiplet.

2.2. Obtaining the Yang–Mills–Higgs theory

To construct the actual Yang–Mills–Higgs theory, one starts with the fixed (for convenience flat) Dirac operator of a four-dimensional spacetime with a fixed fermionic mass matrix. To generate curvature, a general coordinate transformation is performed and then the Dirac operator is fluctuated. This can be achieved by lifting the automorphisms of the algebra to the Hilbert space, unitarily transforming the Dirac operator with the lifted automorphisms and then building linear combinations. Again, it is sufficient to restrict the treatment to the finite case.

All the automorphisms of matrix algebras connected to the unity element, $\mathrm{Aut}(\mathcal{A})^e$, are inner, i.e. they are of the form

$$i_u a = uau^*, \qquad a \in \mathcal{A}, \tag{3}$$

where

$$u \in \mathcal{U}(\mathcal{A}) = \{u \in \mathcal{A} \mid u^*u = uu^* = 1\} \tag{4}$$

is an element of the group of unitaries of the algebra and i is a map from the unitaries into the inner automorphisms $\mathrm{Int}(\mathcal{A})$:

$$i : \mathcal{U}(\mathcal{A}) \longrightarrow \mathrm{Int}(\mathcal{A}), \qquad u \longmapsto i_u. \tag{5}$$

In the kernel of i are the central unitaries, which commute with all elements in $\mathcal{A}$. These inner automorphisms $\mathrm{Int}(\mathcal{A})$ are equivalent to the group of unitaries $\mathcal{U}(\mathcal{A})$ modulo the central unitaries $\mathcal{U}^c(\mathcal{A})$.

The Abelian algebras $\mathbb{R}$ and $\mathbb{C}$ do not possess any inner automorphisms. Remarkably, the quaternions and the matrix algebras over the complex numbers produce the kind of inner automorphisms that correspond to the non-Abelian gauge groups of the standard model. Note that the exceptional groups do not appear. They are the automorphism groups of non-associative algebras.

As in the Riemannian case, the automorphisms close to the identity are going to be lifted to the Hilbert space. This lift has a simple closed form [6], $L = \hat{L} \circ i^{-1}$ with

$$\hat{L}(u) = \rho(u) J \rho(u) J^{-1}. \tag{6}$$

Here two crucial problems occur. The symmetry group of the standard model contains an Abelian sub-group $U(1)_Y$. But the inner automorphisms do not contain any Abelian sub-groups by definition. Furthermore, the lift is multivalued for matrix algebras over the complex numbers since the kernel of i contains a $U(1)$ group. Note that neither the matrix algebras over the reals nor those over the quaternions have any central unitaries close to the identity. The solution to both of these problems is to centrally extend the lift, i.e. to adjoin some central elements [11]. One has to distinguish between central unitaries stemming from the Abelian algebra $\mathbb{C}$ and those from non-Abelian matrix algebras $M_n(\mathbb{C})$, $n \geqslant 2$. To simplify, let the algebra $\mathcal{A}$ be a sum of matrix algebras over the complex numbers. Furthermore, the commutative and noncommutative sub-algebras will be separated,

$$\mathcal{A} = \mathbb{C}^M \oplus \bigoplus_{k=1}^{N} M_{n_k}(\mathbb{C}) \ni (b_1, \ldots, b_M, c_1, \ldots, c_N), \qquad n_k \geqslant 2. \tag{7}$$

The group of unitaries $\mathcal{U}(\mathcal{A})$ and the group of central unitaries $\mathcal{U}^c(\mathcal{A})$ are then given by

$$\begin{aligned} \mathcal{U}(\mathcal{A}_f) &= U(1)^M \times U(n_1) \times \cdots \times U(n_N) \ni u = (v_1, \ldots, v_M, w_1, \ldots, w_N), \\ \mathcal{U}^c(\mathcal{A}_f) &= U(1)^{M+N} \ni u^c = \left(v_1, \ldots, v_M, w_1^c 1_{n_1}, \ldots, w_N^c 1_{n_N}\right). \end{aligned} \tag{8}$$

For the inner automorphisms follows

$$\mathrm{Int}(\mathcal{A}) = \mathcal{U}(\mathcal{A})/\mathcal{U}^c(\mathcal{A}) \ni u^{\mathrm{in}} = \big(1, \ldots, 1, w_1^{\mathrm{in}}, \ldots, w_N^{\mathrm{in}}\big), \tag{9}$$

with $w_j^{\mathrm{in}} \in \mathcal{U}\big(M_{n_j}\big)\big/U(1)$. The lift $L = \hat{L} \circ i^{-1}$ can be written explicitly with

$$\hat{L} = \rho(1, \ldots, 1, w_1, \ldots, w_M) J\rho(\cdots)J^{-1}. \tag{10}$$

It is multivalued due to the kernel of i, $\ker(i) = \mathcal{U}^c(\mathcal{A}_f)$. This multivaluedness can be cured by introducing an additional lift ℓ for the central unitaries, which is restricted to those unitaries $\mathcal{U}^{\mathrm{nc}}(\mathcal{A})$ stemming from the noncommutative part of the algebra,

$$\begin{aligned}\ell\big(w_1^c, \ldots, w_N^c\big) := \rho\Bigg(&\prod_{j_1=1}^{N} \big(w_{j_1}^c\big)^{q_{1,j_1}}, \ldots, \prod_{j_M=1}^{N} \big(w_{j_M}^c\big)^{q_{M,j_M}}, \prod_{j_{M+1}=1}^{N} \big(w_{j_{M+1}}^c\big)^{q_{1,j_{M+1}}} 1_{n_1}\\ &\ldots, \prod_{j_{M+N}=1}^{N} \big(w_{j_{M+N}}^c\big)^{q_{1,j_{M+N}}} 1_{n_N}\Bigg) J\rho(\cdots)J^{-1},\end{aligned} \tag{11}$$

with the $(M + N) \times N$ matrix of charges $q_{k,j}$. The extended lift $\mathbb{L}$ is then defined as

$$\mathbb{L}(u^i, w^c) := (\hat{L} \circ i^{-1})(u^i)\ell(w^c), \qquad u^i \in \mathrm{Int}(\mathcal{A}), \quad w^c \in \mathcal{U}^{\mathrm{nc}}(\mathcal{A}). \tag{12}$$

For convenience, this lift will be written as $\mathbb{L}(u)$ without making the specific distinction between the unitaries and the central unitaries.

In this way, Abelian gauge groups have been introduced and the multivaluedness has been reduced, depending on the choice of the matrix of charges.

The *fluctuation* ${}^f\mathcal{D}$ of the Dirac operator $\mathcal{D}$ is given by a finite collection f of real numbers r_j and algebra automorphisms $u_j \in \mathrm{Aut}(\mathcal{A})^e$ such that

$${}^f\mathcal{D} := \sum_j r_j \mathbb{L}(u_j)\mathcal{D}\mathbb{L}(u_j)^{-1}, \qquad r_j \in \mathbb{R}, \quad u_j \in \mathrm{Aut}(\mathcal{A})^e.$$

These fluctuated Dirac operators build a vector space which serves as the configuration space for the Yang–Mills–Higgs theory. Only fluctuations with real coefficients are considered since ${}^f\mathcal{D}$ must remain self-adjoint.

The submatrix of the fluctuated Dirac operator ${}^f\mathcal{D}$ which is equivalent to the mass matrix $\mathcal{M}$ is often denoted by φ, the 'Higgs scalar', in the physics literature. But one has to be careful, as will be shown below explicitly. It may happen that the lifted automorphisms or parts of it commute with the initial Dirac operator and one finds ${}^f\mathcal{D} = \sum_i r_i\mathcal{D}$ for the finite part of the spectral triple. This behaviour appeared for the first time in the electro-strong model in [4], where the fermions couple vectorially to all gauge groups and no Higgs field appears. In the model presented below, the spectral triple can be decomposed into a direct sum consisting of the standard model and two new particles. The initial Dirac operator of the new particles commutes with the corresponding part of the lift and thus does not participate in the Higgs mechanism.

As mentioned in the introduction, an almost-commutative geometry is the tensor product of a finite noncommutative triple with an infinite commutative spectral triple. By Connes' reconstruction theorem [7], it is known that the latter comes from a Riemannian spin manifold, which will be taken to be any four-dimensional, compact, flat manifold such as the flat 4-torus. The spectral action of this almost-commutative spectral triple is computed from the heat-kernel expansion and then reduced to the finite part. It is a functional on the vector space of all fluctuated, finite Dirac operators:

$$V({}^f\mathcal{D}) = \lambda\, \mathrm{tr}[({}^f\mathcal{D})^4] - \frac{\mu^2}{2}\, \mathrm{tr}[({}^f\mathcal{D})^2],$$

where λ and μ are positive constants [9]. The spectral action is invariant under lifted automorphisms and even under the unitary group $U(\mathcal{A}) \ni u$,

$$V([\rho(u)J\rho(u)J^{-1}]^{f}\mathcal{D}[\rho(u)J\rho(u)J^{-1}]^{-1}) = V(^{f}\mathcal{D}),$$

and it is bounded from below. To obtain the physical content of a diagram and its associated spectral triple, one has to find the minima $^{\hat{f}}\mathcal{D}$ of this action with respect to the lifted automorphisms and their spectra.

2.3. Irreducibility, non-degeneracy

For the classification presented in [1–4], some extra conditions were imposed onto the almost-commutative spectral triples. The spectral triples were required to be irreducible and non-degenerate according to the following definitions.

Definition 1.

(1) A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is degenerate if the kernel of $\mathcal{D}$ contains a non-trivial subspace of the complex Hilbert space $\mathcal{H}$ invariant under the representation ρ on $\mathcal{H}$ of the real algebra $\mathcal{A}$.

(2) A non-degenerate spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is reducible if there is a proper subspace $\mathcal{H}_0 \subset \mathcal{H}$ invariant under the algebra $\rho(\mathcal{A})$ such that $(\mathcal{A}, \mathcal{H}_0, \mathcal{D}|_{\mathcal{H}_0})$ is a non-degenerate spectral triple. If the triple is real, S^0-real and even, it is required that the subspace $\mathcal{H}_0$ is also invariant under the real structure J, the S^0-real structure ϵ and under the chirality χ such that the triple $(\mathcal{A}, \mathcal{H}_0, \mathcal{D}|_{\mathcal{H}_0})$ is again real, S^0-real and even.

Definition 2. *The irreducible spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is dynamically non-degenerate if all minima $^{\hat{f}}\mathcal{D}$ of the action $V(^{f}\mathcal{D})$ define a non-degenerate spectral triple $(\mathcal{A}, \mathcal{H}, {}^{\hat{f}}\mathcal{D})$ and if the spectra of all minima have no degeneracies other than the three kinematical degeneracies: left–right, particle–antiparticle and colour. The definition of 'colour' will be given in section 2.4. Of course, in the massless case there is no left–right degeneracy. It is also supposed that the colour degeneracies are protected by the little group. This means that all eigenvectors of $^{\hat{f}}\mathcal{D}$ corresponding to the same eigenvalue are in a common orbit of the little group (and scalar multiplication and charge conjugation).*

In physicists' language, non-degeneracy excludes all models with pairwise equal fermion masses in the left-handed particle sector up to colour degeneracy. Irreducibility means the restriction to one fermion generation and to keep the number of fermions as small as allowed by the axioms for spectral triples. The last requirement of definition 2 means that noncommutative colour groups are unbroken. It ensures that the corresponding mass degeneracies are protected from quantum corrections. It should be noted that the standard model of particle physics meets all these requirements.

2.4. Krajewski diagrams

Connes' axioms, the decomposition of the Hilbert space, the representation and the Dirac operator allow a diagrammatic depiction. As was shown in [1, 12], this can be boiled down to simple arrows, which encode the multiplicity matrix and the fermionic mass matrix. From this information, all the ingredients of the spectral triple can be recovered. For the present purpose, a simple arrow and connections of arrows at one point are sufficient. The arrows always point from right fermions (positive chirality) to left fermions (negative chirality). It suffices to restrict the diagram to the particle part, since the information of the antiparticle part is included by transposing the particle part. The conventions of [1] will be used so that algebra elements tensorized with $1_{m_{ij}}$ will be written as a direct sum of m_{ij} summands.

• The Dirac operator: the components of the (internal) Dirac operator are represented by horizontal or vertical lines connecting two nonvanishing entries of opposite signs in the multiplicity matrix μ. For the purpose of this paper, the simple horizontal arrows are sufficient, so only these will be treated in this section.

Each arrow represents a nonvanishing, complex submatrix in the Dirac operator. For instance, μ_{ij} can be linked to μ_{ik} or μ_{kj} by

$$\begin{array}{ccc} \circ & \longleftarrow & \circ \\ \mu_{ij} & & \mu_{ik} \end{array}$$

and these arrows represent a submatrix of $\mathcal{M}$ in $\mathcal{D}$ of type $M \otimes 1_{(n_i)}$ with M a complex $(n_j) \times (n_k)$ matrix.

The requirement of non-degeneracy of a spectral triple means that every nonvanishing entry in the multiplicity matrix μ is touched by at least one arrow.

More complicated connected arrows appear when treating the standard model and when classifying almost-commutative spectral triples. But since the standard model appears as a disconnected sub-model in this paper, and thus is not treated in detail, the reader is referred to [1] for a more detailed treatment.

• Convention for the diagrams: a *simple arrow* will connect +1 to −1, that is, right chirality to left chirality:

$$\begin{array}{ccc} \circ & \longleftarrow & \circ \\ -1 & & +1 \end{array}$$

For a given algebra, every spectral triple is encoded in its multiplicity matrix which itself is encoded in its Krajewski diagram, a field of arrows. In these conventions, for particles, $\epsilon = 1$, the column label of the multiplicity matrix indicates the representation, the row label indicates the multiplicity. For antiparticles, the row label of the multiplicity matrix indicates the representation, the column label indicates the multiplicity.

The *circles* in the diagrams only intend to guide the eye. According to these rules, the numbers ± 1 under the arrows can be omitted, since they are now redundant.

Every arrow comes with three algebras: two algebras that localize its end points, let us call them *right and left algebras*, and a third algebra that localizes the arrow, let us call it *colour algebra*. For example, for the arrow

$$\begin{array}{ccc} \circ & \longleftarrow & \circ \\ \mu_{ij} & & \mu_{ik} \end{array}$$

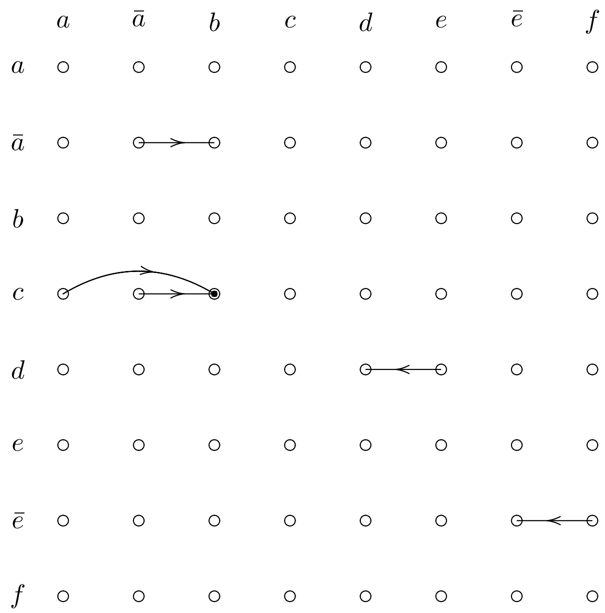
the left algebra is $\mathcal{A}_j$, the right algebra is $\mathcal{A}_k$ and the colour algebra is $\mathcal{A}_i$.

For an extensive list of examples how to translate between Krajewski diagrams and the corresponding spectral triples, the reader is again referred to [1].

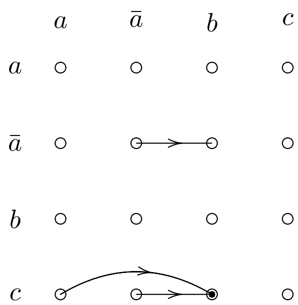
It can be shown that irreducible spectral triples are depicted by minimal (i.e. irreducible) Krajewski diagrams. These can be intuitively understood as diagrams which have as few arrows as possible. If one removed any number of arrows from a minimal Krajewski diagram, the determinant of the corresponding multiplicity matrix would become zero. Consequently, the axiom of the Poincaré duality would be violated.

3. The spectral triple

The Krajewski diagram of the counter-example encodes an almost-commutative spectral triple with six summands in the internal algebra:



One can clearly see the sub-diagram of the standard model



in the upper 4×4 corner. The multiplicity matrix of the complete Krajewski diagram is readily found to be

$$\mu = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{13}$$

which confirms due to $\det \cap = \det(\mu + \mu^T) = 4$, the Poincaré duality.

Essentially, the diagram is a direct sum of diagrams (1) and (17) given in [1]. The matrix is already *blown up* in the sense that the representations of the complex parts of the matrix algebra

$$\mathcal{A} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \ni (a, b, c, d, e, f)$$

are fixed:

$$\begin{aligned}
\rho_L(a,b,c,d,e,f) &= \begin{pmatrix} b\otimes 1_3 & 0 & 0 & 0\\ 0 & b & 0 & 0\\ 0 & 0 & d & 0\\ 0 & 0 & 0 & \bar{e}\end{pmatrix},\\
\rho_R(a,b,c,d,e,f) &= \begin{pmatrix} a1_3 & 0 & 0 & 0 & 0\\ 0 & \bar{a}1_3 & 0 & 0 & 0\\ 0 & 0 & \bar{a} & 0 & 0\\ 0 & 0 & 0 & e & 0\\ 0 & 0 & 0 & 0 & f\end{pmatrix},\\
\rho_L^c(a,b,c,d,e,f) &= \begin{pmatrix} 1_2\otimes c & 0 & 0 & 0\\ 0 & \bar{a}1_2 & 0 & 0\\ 0 & 0 & d & 0\\ 0 & 0 & 0 & \bar{e}\end{pmatrix},\\
\rho_R^c(a,b,c,d,e,f) &= \begin{pmatrix} c & 0 & 0 & 0 & 0\\ 0 & c & 0 & 0 & 0\\ 0 & 0 & \bar{a} & 0 & 0\\ 0 & 0 & 0 & e & 0\\ 0 & 0 & 0 & 0 & \bar{e}\end{pmatrix}.
\end{aligned} \tag{14}$$

These representations are faithful on the Hilbert space given below and serve as well to construct the lift of the automorphism group. As pointed out in [11], the commutative sub-algebras of $\mathcal{A}$ which are equivalent to the complex numbers serve as receptacles for the $U(1)$ sub-groups embedded in the automorphism group $U(3)$ of the $M_3(\mathbb{C})$ matrix algebra. In this way, the standard model gauge groups $U(1)_Y$ and $SU(3)_C$ are generated. Roughly speaking, each diagonal entry of the representation of the algebra can be associated with fermion multiplet. For example, the first entry of ρ_L, $b\otimes 1_3$, is the representation of the algebra on the up and down quark doublet, where each quark is again a colour triplet.

The algebra $\mathcal{A}$ is chosen to reproduce the standard model gauge group and to respect the physical requirements proposed in [1]. One finds that to the standard model algebra

$$\mathcal{A}_{\mathrm{SM}} = \mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C}) \ni (a,b,c) \tag{15}$$

only three copies of the complex numbers may be added. Any other matrix algebra would either lead to a broken colour group or dynamically degenerate fermion masses, i.e. a second massless, neutrino-like particle.

It is now straightforward that the algebra as well as its representation splits into direct sums of the usual standard model algebra and its representation, as well as an algebra and a representation for the new particles:

$$\mathcal{A} = \mathcal{A}_{\mathrm{SM}}\oplus\mathcal{A}_{\mathrm{new}} \tag{16}$$

with $\mathcal{A}_{\mathrm{new}} = \mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}\ni(d,e,f)$ and

$$\rho = \rho_{L,\mathrm{SM}}\oplus\rho_{L,\mathrm{new}}\oplus\rho_{R,\mathrm{SM}}\oplus\rho_{R,\mathrm{new}}\oplus\overline{\rho}^c_{L,\mathrm{SM}}\oplus\overline{\rho}^c_{L,\mathrm{new}}\oplus\overline{\rho}^c_{R,\mathrm{SM}}\oplus\overline{\rho}^c_{R,\mathrm{new}}. \tag{17}$$

The new parts of the representation can be read off from (14) and are given by

$$\begin{aligned}
\rho_{L,\mathrm{new}}(d,e,f) &= \begin{pmatrix} d & 0\\ 0 & \bar{e}\end{pmatrix}, \qquad & \rho_{R,\mathrm{new}}(d,e,f) &= \begin{pmatrix} e & 0\\ 0 & f\end{pmatrix},\\
\rho^c_{L,\mathrm{new}}(d,e,f) &= \begin{pmatrix} d & 0\\ 0 & \bar{e}\end{pmatrix}, \qquad & \rho^c_{R,\mathrm{new}}(d,e,f) &= \begin{pmatrix} e & 0\\ 0 & \bar{e}\end{pmatrix}.
\end{aligned} \tag{18}$$

The Hilbert space is given as well by the direct sum of the standard model Hilbert space, for details see [10], and a Hilbert space containing two new fermions

$$\mathcal{H} = \mathcal{H}_{\rm SM} \oplus \mathcal{H}_{\rm new},$$

where

$$\mathcal{H}_{\rm new} \ni \begin{pmatrix} \psi_{1L} \\ \psi_{2L} \end{pmatrix} \oplus \begin{pmatrix} \psi_{1R} \\ \psi_{2R} \end{pmatrix} \oplus \begin{pmatrix} \psi_{1L}^c \\ \psi_{2L}^c \end{pmatrix} \oplus \begin{pmatrix} \psi_{1R}^c \\ \psi_{2R}^c \end{pmatrix}.$$

The wavefunctions $\psi_{1L}, \psi_{2L}, \psi_{1R}$ and ψ_{2R} are the respective left- and right-handed Dirac 4-spinors. One finds for the real structure, the chirality and the S^0-reality

$$J = \begin{pmatrix} 0 & 1_{\dim \mathcal{H}/2} \\ 1_{\dim \mathcal{H}/2} & 0 \end{pmatrix} \circ \text{complex conjugation}, \tag{19}$$

$$\chi = \mathrm{diag}\big(-1_{\dim \mathcal{H}_L}, 1_{\dim \mathcal{H}_R}, -1_{\dim \mathcal{H}_L}, 1_{\dim \mathcal{H}_R}\big) \tag{20}$$

and

$$\epsilon = \mathrm{diag}\big(1_{\dim \mathcal{H}_L}, 1_{\dim \mathcal{H}_R}, -1_{\dim \mathcal{H}_L}, -1_{\dim \mathcal{H}_R}\big), \tag{21}$$

where the subscripts L and R denote the left- and right-handed parts of the Hilbert space, respectively.

The initial internal Dirac operator, which is to be fluctuated with the lifted automorphisms, is chosen to have as its mass matrix

$$\mathcal{M} = \begin{pmatrix} \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \otimes 1_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_e & 0 & 0 \\ 0 & 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & 0 & m_2 \end{pmatrix} = \mathcal{M}_{\rm SM} \oplus \mathcal{M}_{\rm new},$$

with $m_u, m_d, m_e, m_1, m_2 \in \mathbb{C}$,

$$\mathcal{M}_{\rm SM} = \begin{pmatrix} \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \otimes 1_3 & 0 & 0 \\ 0 & 0 & m_e \end{pmatrix} \qquad \text{and} \qquad \mathcal{M}_{\rm new} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}. \tag{22}$$

Here again the general structure of a direct sum appears. But the physical model cannot be considered as a direct sum of the standard model and the new particles, since the gauge group for the new particles, which is generated by the lift, has its origin in the $M_3(\mathbb{C})$ summand of the standard model algebra.

It should be pointed out that the above choice for the summands of the matrix algebra, the Hilbert space and the Dirac operator is rather unique, if one requires the Hilbert space to be minimal and the fermion masses to be non-degenerate.

For the following calculations, it is sufficient to consider the particle part of the lift for the automorphisms. The non-Abelian unitaries of $\mathcal{A}$ close to the identity are $\mathcal{U}^e(\mathcal{A}) = SU(2) \times U(3)$ and the corresponding central unitaries are simply the determinants of the elements in $U(3)$ since the determinant of each element in $SU(2)$ is unity. These central unitaries form the $U(1)$ gauge group of the standard model.

Using the definition of the extended lift (12) and the real structure (19), one finds with $w^c = \det w$ for the particle part of the lift

$$\begin{aligned} \mathbb{L}_P((w^c)^p, u, (w^c)^q w, (w^c)^r, (w^c)^s, (w^c)^t) &= \rho_L((w^c)^p, u, (w^c)^q w, (w^c)^r, (w^c)^s, (w^c)^t) \\ &\quad \times \rho_L^c((w^c)^p, u, (w^c)^q w, (w^c)^r, (w^c)^s, (w^c)^t) \\ &\quad \oplus \rho_R((w^c)^p, u, (w^c)^q w, (w^c)^r, (w^c)^s, (w^c)^t) \end{aligned}$$

$$\times \rho_R^c((w^c)^p, u, (w^c)^q w, (w^c)^r, (w^c)^s, (w^c)^t)$$
$$= \mathrm{diag}\left[(w^c)^q u \otimes w, (w^c)^{-p} u, (w^c)^{2r}, (w^c)^{-2s}\right.$$
$$\left.(w^c)^{p+q} w, (w^c)^{q-p} w, (w^c)^{-2p}, (w^c)^{2s}, (w^c)^{t-s}\right], \tag{23}$$

where $u \in SU(2)$, $w \in U(3)$ and the hypercharges $p, q, r, s, t \in \mathbb{Q}$. This lift decomposes again into a sum with parts for the standard model and the new particles,

$$\mathbb{L}_P = \mathbb{L}_{L,\mathrm{SM}} \oplus \mathbb{L}_{L,\mathrm{new}} \oplus \mathbb{L}_{R,\mathrm{SM}} \oplus \mathbb{L}_{R,\mathrm{new}}, \tag{24}$$

where the parts of the lift for the new particles are given by

$$\mathbb{L}_{L,\mathrm{new}}((w^c)^r, (w^c)^s, (w^c)^t) = \mathrm{diag}\,[(w^c)^{2r}, (w^c)^{-2s}] \tag{25}$$

and

$$\mathbb{L}_{R,\mathrm{new}}((w^c)^r, (w^c)^s, (w^c)^t) = \mathrm{diag}\,[(w^c)^{2s}, (w^c)^{t-s}]. \tag{26}$$

The conditions for this lift being free of harmful anomalies reduce for the standard model part to the relation [11]

$$q = \frac{p-1}{3}. \tag{27}$$

This also leads to the reduction $U(3) \to SU(3)$ for the colour gauge group due to the central extension $(\det w)^q$ in the lift. The new particles couple only to the $U(1)$ part of the gauge group and thus are free of harmful anomalies if this coupling is vectorial. It follows that

$$r = s \qquad \text{and} \qquad t = -s, \tag{28}$$

i.e. left- and right-handed particles have equal $U(1)$ charges and the two new particles possess equal but opposite charges.

Fluctuating the Dirac operator leads for the standard model part to the usual Higgs potential

$$\varphi = \sum_i r_i \mathbb{L}_{L,\mathrm{SM}}\big((w_i^c)^p, u_i, (w_i^c)^q w_i\big)\mathcal{M}_{\mathrm{SM}}\big(\mathbb{L}_{R,\mathrm{SM}}\big((w_i^c)^p, u_i, (w^c)_i^q w_i\big)\big)^{-1}. \tag{29}$$

For the new particles on the other hand the lift is a diagonal matrix and thus commutes with the corresponding diagonal mass matrix. And for the choice of vectorial couplings, one finds

$$\mathbb{L}_{L,\mathrm{new}}((w^c)^r, (w^c)^r, (w^c)^{-r})\mathcal{M}_{\mathrm{new}}(\mathbb{L}_{R,\mathrm{new}}((w^c)^r, (w^c)^r, (w^c)^{-r}))^{-1} = \mathcal{M}_{\mathrm{new}}. \tag{30}$$

Therefore, the new particles do not participate in the Higgs mechanism but acquire their masses through the Dirac operator directly from the geometry. These masses do not break gauge invariance, since they couple vectorially to the Abelian $U(1)$ sub-group of the standard model, only. To conform to standard notation, the hypercharges q and r will be denoted by Y with appropriate normalization for the corresponding particle.

Calculating the spectral action gives the usual Einstein–Hilbert action, the Yang–Mills–Higgs action of the standard model and a new part in the Lagrangian for the two new fermions:

$$\mathcal{L}_{\mathrm{new}} = \mathrm{i}\psi_{1L}^* D_1 \psi_{1L} + \mathrm{i}\psi_{1R}^* D_1 \psi_{1R} + \tilde{m}_1 \psi_{1L}^* \psi_{1R} + \tilde{m}_1 \psi_{1R}^* \psi_{1L} + \mathrm{i}\psi_{2L}^* D_2 \psi_{2L} + \mathrm{i}\psi_{2R}^* D_2 \psi_{2R}$$
$$+ \tilde{m}_2 \psi_{2L}^* \psi_{2R} + \tilde{m}_2 \psi_{2R}^* \psi_{2L},$$

with $\tilde{m}_{1/2} = n m_{1/2}$, where the factor $n \in \mathbb{R}^+$ is determined by the coefficients r_i from the Higgs potential (29). For details of this rather longish calculation see [9, 11]. The covariant derivative couples the two fermions to the $U(1)$ sub-group of the standard model gauge group,

$$\begin{aligned} D_{1/2} &= \gamma^\mu \partial_\mu + \frac{\mathrm{i}}{2} g' Y_{1/2} \gamma^\mu B_\mu \\ &= \gamma^\mu \partial_\mu + \frac{\mathrm{i}}{2} e Y_{1/2} \gamma^\mu A_\mu - \frac{\mathrm{i}}{2} g' \sin\theta_w Y_{1/2} \gamma^\mu Z_\mu, \end{aligned}$$

where B is the gauge field corresponding to $U(1)$, A and Z are the photon and the Z-boson fields, e is the electro-magnetic coupling and θ_w is the weak angle. The hypercharge $Y_{1/2}$ of the new fermions can be any non-zero fractional number with $Y_1 = -Y_2$ so that ψ_1 and ψ_2 have opposite electrical charge. $Y_1 = 0$ is not allowed since this would conflict with the postulated complex representation of the little group on the fermions. For simplicity, $Y_1 = 1$ may be chosen which results in opposite electro-magnetic charges $\pm e$ for ψ_1 and ψ_2.

4. Speculations on dark matter

The question which arises is whether these new particles could be realized in nature. It is tempting to identify them with the missing dark matter, but the apparent electro-magnetic charge seems to pose a problem. One could though speculate that dark matter does not consist of elementary particles but instead of hydrogen-like composite particles built out of the oppositely charged new fermions. Assuming that the mass m_1 of the lighter particle is of the order of 100 GeV or higher and $m_2 \geqslant m_1$, the composite two-particle system would have a radius comparable to a neutron or even smaller. This estimation of the mass has been chosen such that no conflict with current accelerator experiments should occur.

The ionization energy of the two-particle system would be of the order of a few MeV, comparable to gamma ray energies of radioactive nuclei. If one further assumes a dark matter density of $\rho_{\mathrm{DM}} \sim 0.5\,\mathrm{TeV\ m^{-3}}$ in the galactic plane, this would result in a particle density of $\rho_{\mathrm{part}} \leqslant 0.5$ particles m^{-3}. These composite particles would freeze out in the early universe when the temperature drops below the ionization energy, in analogy with the freeze-out of hydrogen in the period of recombination.

Since these hydrogen-like particles are electro-magnetically neutral and do not interact strongly, it would be very difficult to detect them directly. It remains of course to investigate whether this kind of dark matter is compatible with the established cosmological models, especially baryogenesis, and whether it is already possible to exclude them with current experiments.

5. Conclusions

A counter-example to the conjecture that almost-commutative geometries, with respect to sensible physical constraints, allow only the standard model of particle physics and electro-strong models as Yang–Mills–Higgs theories, given in [1–4], has been presented. The corresponding Yang–Mills–Higgs model predicts two new fermions with opposite electro-magnetic charge and weak interaction through the Z-boson. Their masses are not acquired via the Higgs mechanism but are Dirac masses, which can be chosen arbitrarily. Combining these particles into a hydrogen-like composite particle could provide a new candidate for dark matter, if compatible with current experiments. Further investigations of these composite particles are clearly necessary. It should be pointed out that the mathematical requirements on spectral triples are extremely tight and that the physical requirements presented above are most general. It is thus highly non-trivial that Yang–Mills–Higgs models containing the standard model of particle physics exist at all in the context of almost-commutative geometry.

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Note added. As could be shown in [15], the above model provides for a realistic candidate for dark matter, if the hypercharge $Y_{1/2}$ is chosen to be ± 2. Further investigations [16] have shown that the lower limit of the masses for these doubly charged fermions is ~ 10 GeV. These results were obtained from high-precision measurements of the anomalous magnetic moment of the muon.

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